BCZ map (F. Boca, C. Cobeli, A. Zaharescu) 2001

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x+y>1,\ x \le 1,\ y \le 1\}$$

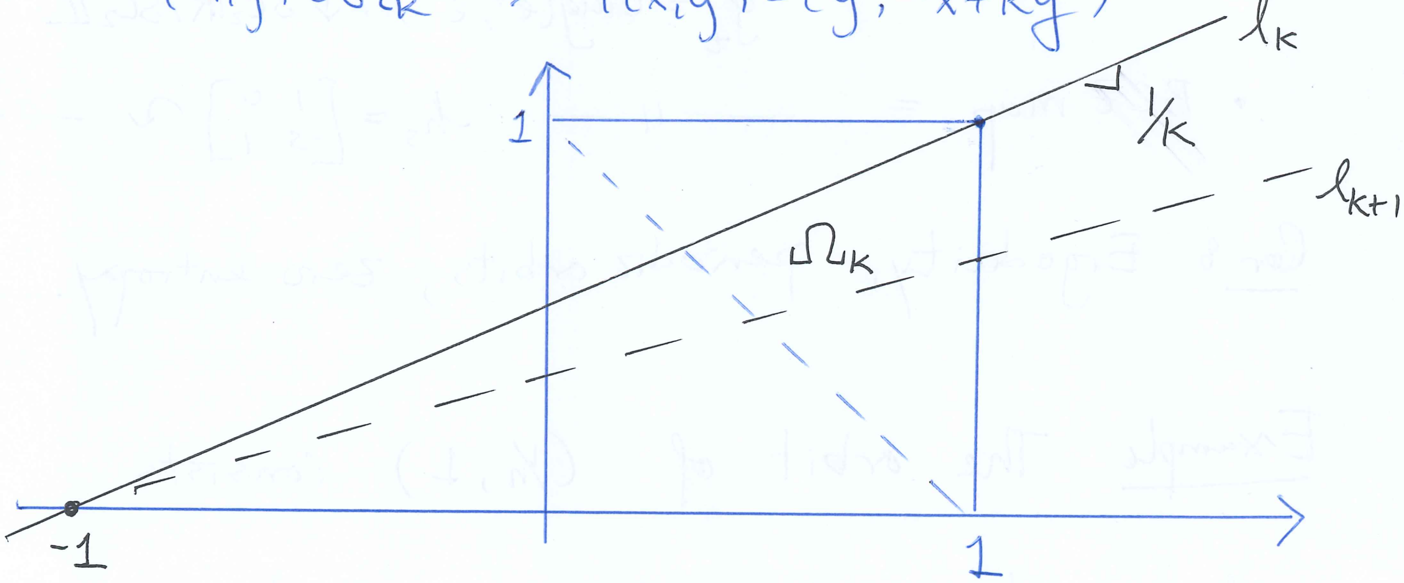
$$= \bigsqcup_{1}^{\infty} \Omega_k \quad (\text{disjoint})$$

Ω_k = level set of $k : \Omega \to \mathbb{Z}_+$

$$k = \left[\frac{1+x}{y}\right]$$

$$\boxed{T\big|_{\Omega_k} = \begin{bmatrix} 0 & 1 \\ -1 & k \end{bmatrix}}$$

$(x,y) \in \Omega_k \Rightarrow T(x,y) = (y, -x+ky)$



T is a bijection : $\boxed{T(\Omega_k) = \Omega_k^T}$ (reflection in $y=x$)

(T, Ω, μ) piecewise linear, area-preserving

μ = normalized Lebesgue

Introduced by Boca-Cobelli-Zaharescu in 2001 to study statistical properties of Farey sequences.

(Athreya-C, 2009) Recognized BCZ map is the analog of Gauss map for horocycle flow.

* Gauss map = Poincare section of geodesic flow on (unit tangent bundle) of the modular surface, (space of lattices)

$$g_t = \mathrm{diag}(e^t, e^{-t}) \curvearrowright SL_2\mathbb{R}/SL_2\mathbb{Z}$$

* BCZ map = —— " —— $h_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \curvearrowright$ —"—.

<u>Cor</u>: Ergodicity, periodic orbits, zero entropy.

<u>Example</u> The orbit of $(1/n, 1)$ consists of points $\left(\frac{a_j}{n}, \frac{b_j}{n}\right)$ s.t. $a_j + b_j > n$, $\gcd(a_j, b_j) = 1$.

$j = 1, \dots, A = \phi(1) + \phi(2) + \dots + \phi(n)$.

<u>Note</u>: $b_1 = n$, b_j = height of P_j $(= a_{j+1})$

Farey sequence of level n

Order rationals in $[0,1] \cap \mathbb{Q}_{\leq n}$ by

$$0 = \rho_0 < \underset{\frac{1}{n}}{\rho_1} < \cdots < \rho_A = 1$$

where $A = A(n) = \phi(1) + \phi(2) + \cdots + \phi(n)$.

(Franel-Landau, 1924)

$$\text{R.H.} \iff \sum_1^A \left(\rho_i - \frac{i}{A}\right)^2 = O\left(\frac{1}{n^{1-\varepsilon}}\right)$$

$$\iff \sum_1^A \left|\rho_i - \frac{i}{A}\right| = O\left(n^{\frac{1}{2}+\varepsilon}\right)$$

(Hall, 1970) The gaps $n^2(\rho_i - \rho_{i-1})$ distribute according to

mean $= \frac{\pi^2}{3}$

no variance

(Boca - Cobeli - Zaharescu, 2001)

- introduced PL-map to ~~[illegible]~~ generate order statistics

__Note:__ ρ_{j+1} is determined by knowledge of n, ρ_j & ρ_{j-1}

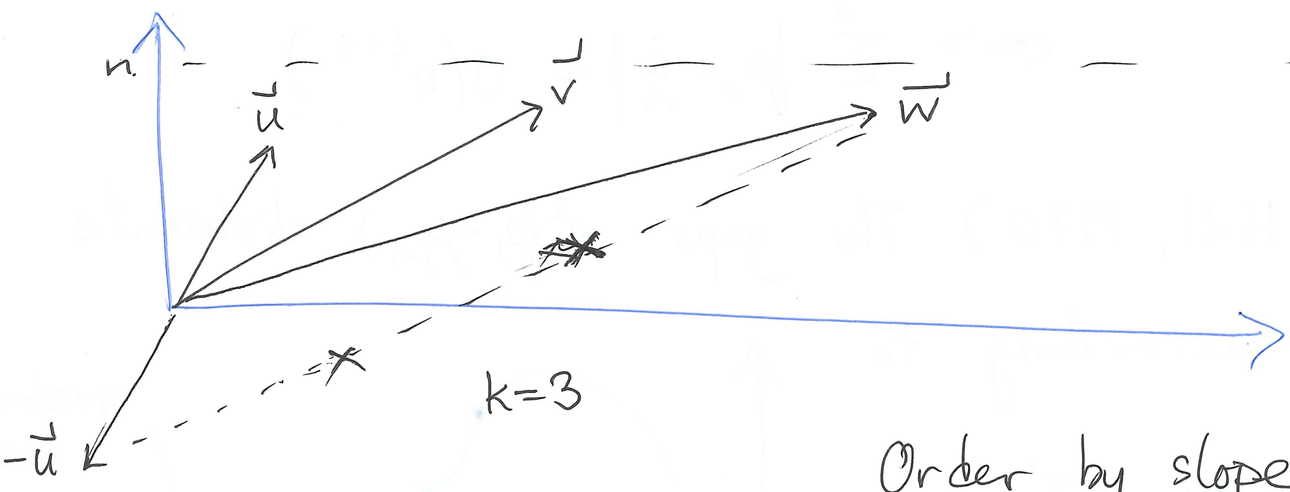
$\frac{p}{q} \in \mathbb{Q}$ lowest terms $\gcd(p,q)=1$, $q>0$ $\longleftrightarrow$ $\vec{v} = \langle p, q \rangle \in \mathbb{Z}^2$

$\left.\begin{array}{l} \rho_{j-1} \longleftrightarrow \vec{u} \\ \rho_j \longleftrightarrow \vec{v} \end{array}\right\}$ $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v} = \mathbb{Z}^2$ integral basis

$\rho_{j+1} \longleftrightarrow \vec{w}$ $\quad \vec{w} = k\vec{v} - \vec{u}$ for some $k \in \mathbb{Z}_+$

Require $\boxed{0 < q \le n}$

$\mathbb{Z}^2$



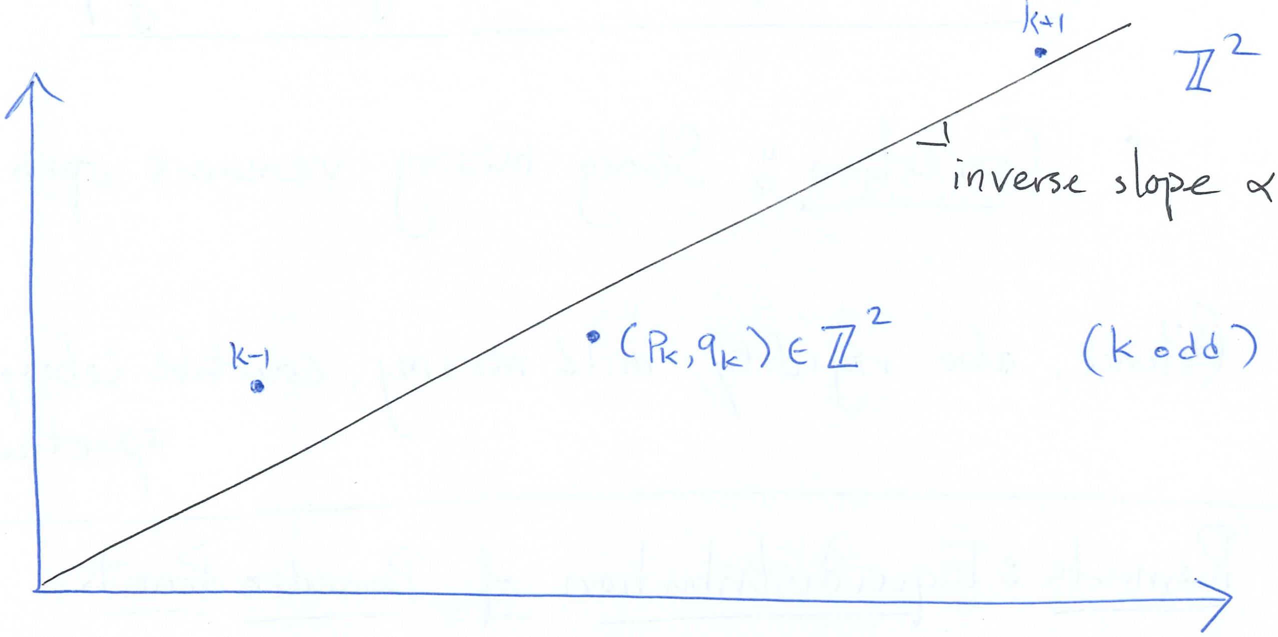
Order by slope.

Instead of fixing the integer lattice, we could fix a direction, say the vertical, and apply a horizontal shear $\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$

(In Athreya-C, we developed the "transpose" picture.)
which is convenient for our preferred view of geodesic flow

Dani Correspondence

Let $\{p_k/q_k\}_{k=0}^{\infty}$ denote convergents of $\alpha \notin \mathbb{Q}$.



Linear action of $g_t = \mathrm{diag}(e^t, e^{-t}) \curvearrowright$ space of lattices

is applied to $\Lambda_\alpha = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2$ sheared integer lattice

$$\left\{ \begin{matrix} \text{best approx} \\ \text{of } 2^{nd} \text{ kind} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{shortest vecter} \\ \text{under } g_t \Lambda_\alpha \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \text{rationals} \\ 0 < p - q\alpha < 1 \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{shortest vecter} \\ \text{under } h_s \Lambda_\alpha \end{matrix} \right\}$$

Ordered by increasing slope.

$$h_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

Anthony Quas, Yiwei Zhang

Moscow Seminar
September 22
2012

Main Result (w/ A. Quas, Y. Zhang)

- $\boxed{\text{BCZ map is weakly mixing}}$

- Correction: Strong mixing remains open.

(Hence, also rigidity, mild mixing, countable Lebesgue spectrum.)

Remark: Equidistribution of Periodic Points

$F \subset \Omega$ finite set

$$\delta_F := \frac{1}{|F|} \sum_{x \in F} \delta_x \qquad \left(\begin{array}{c}\text{purely atomic} \\ \text{supported on } F\end{array}\right)$$

Let F_n = orbit of $(1/n, 1)$.

Theorem (A-C, '09) $\delta_{F_n} \to \mu$ as $n \to \infty$.

Proof Equidistribution of closed horocycles (Sarnak '81) #

- Many statistical properties of Farey sequences follow from this.

- We couldn't prove any new number-theoretic result.

Infinitesimal self-similarity

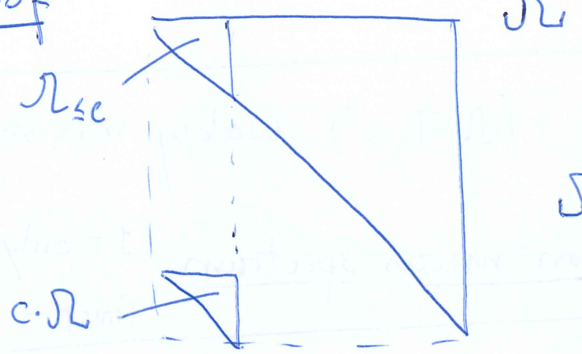
$$\Omega_{\leq c} = \{ (x,y) \in \Omega : x \leq c \} \qquad 0 < c < 1$$

T_c = first return map to $\Omega_{\leq c}$

μ_c = Lebesgue (normalized) on $\Omega_{\leq c}$.

Theorem (Athreya–C '09) $(\Omega_{\leq c}, T_c, \mu_c) \approx (\Omega, T, \mu)$.

Proof



geodesic flow

renormalized horocycle flow

$$\Omega_{\leq c} = c\cdot\Omega \simeq \Omega$$

change of coordinates

$SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ = space of oriented lattices $\Lambda \subset \mathbb{R}^2$ having $\det(\Lambda) = 1$. (unimodular)

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} SL(2,\mathbb{Z}) \longleftrightarrow \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$$

$\boxed{\Omega_{\leq c} \text{ parametrizes lattices with a horizontal vector of length at most } c.}$

Proof of WM requires ability to take limit $c \to 1^-$.

Def (Ω, T, μ) mixing if $\forall\ A, B \in \mathcal{B}(\Omega)$

$$\mu(T^{-n}A \cap B) \to \mu(A)\mu(B) \quad \text{as } n \to \infty.$$

Remark: Mixing means "no memory".

Def (Ω, T, μ) weakly mixing if — " —

$$\frac{1}{n}\sum_0^{n-1} \left| \mu(T^{-k}A \cap B) - \mu(A)\mu(B) \right| \to 0 \quad \text{as } n \to \infty.$$

Theorem (Walters, pp. 48-9) (Ω, T, μ) weakly mixing $\iff$ $U_T : L^2(\mu) \circlearrowleft$ has continuous spectrum (1 = only eigenvalue, Const = — " — eigenfunct.)

$U_T(f) = f \circ T$ Koopman operator

Proof by contradiction ($\nexists$ nonconstant cplx-valued eigenfunctions)

Suppose $Tf = \alpha f$ for some $\alpha \neq 1$.

By ergodicity, may assume $|f| = 1$ a.e.

Let f_c = eigenfunction for $T_c : \Omega_{\leq c} \circlearrowleft$

Extend f_c and T_c trivially to all of Ω.

Observe that $\forall\ 1 \le p < \infty$ (Fix $p=1$ once & for all)

$$\|f - f_c\|_p \to 0 \quad \text{as} \quad c \to 1^-.$$

For any positive integer N,

$$\|f \circ T_c^N - \alpha^N f\| \le 2\|f - f_c\|.$$

$$\left(\le \|f \circ T_c^N - f_c \circ T_c^N\| + \cancel{\|f_c \circ T_c^N - \alpha^N f_c\|}^{\to 0} + \|\alpha^N f_c - \alpha^N f\| \right)$$

All we will need to know (about T_c^N) is the set E of points $\omega \in \Omega$ where

$$\boxed{T_c^N(\omega) = T^{N+1}(\omega)} \qquad (*)$$

which implies $f \circ T_c^N(\omega) = f \circ T^{N+1}(\omega) = \alpha^{N+1} f(\omega)$.

so that $\boxed{\|f \circ T_c^N - \alpha^N f\| \ge |\alpha - 1| \mu(E)}$

Reduced to showing $\mu(E) \ge \delta$ for some absolute δ.

Note: $E = E(c, N)$ $\quad N = N(c) \nearrow \infty$ as $c \to 1^-$

Random collection of lattices

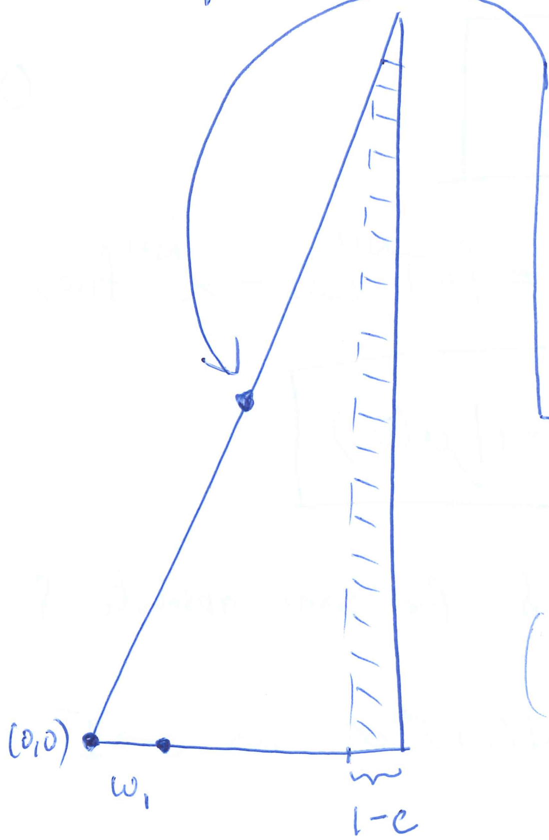
$\omega \in \Omega \rightsquigarrow \Lambda_\omega \subset \mathbb{R}^2$ 2D lattice

$$\Lambda_\omega = \mathbb{Z}\langle \omega_1, 0\rangle + \mathbb{Z}\langle \omega_2, \omega_1^{-1}\rangle$$

Q: Geometric interpretation of $\omega \in E$ or $(*)$?

For convenience, let's fix $\boxed{N_c = \left[\frac{a}{1-c}\right]}$ for some $a > 0$ tbd.

Visualize 2 right triangles in first quadrant, with Λ_ω in background.



slope defined by interior of larger Δ (including $x=1$) having exactly N_c primitive points.

$\omega \in E \Leftrightarrow$ shaded strip has exactly one primitive lattice pt.

$(\Leftrightarrow (*))$

slope = smallest $s > 0$ s.t. $h_s \Lambda_w \in \Omega_1$

where $h_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$ "vertical shear",

= first return time to Ω_1

starting from w.

We know the average return time is $\frac{\pi^2}{3}$.

Ergodic thm + Tchebyshev inequality $\Rightarrow$

on a set of large measure arbitrarily close to 1

we have

$$\frac{R_c}{N_c} \in [3.2, 3.4] \qquad \left(\frac{\pi^2}{3} \simeq 3.29\right)$$

where $R_c = R_c(w)$ = slope of hypotenuse $\left(\begin{matrix} c_0 < c < 1 \\ a > 0 \end{matrix}\right)$

Note: $\boxed{\text{area}(S_c) \asymp a.}$ S_c = shaded strip.

Claim: (i) $\mathbb{P}\text{rob}(\#(\Lambda_w^{pr} \cap S_c) = 1) \gg a$.

Final step
choose a small...

(ii) $\mathbb{P}\text{rob}(\#(\Lambda_w^{pr} \cap B_c) \geq 2) = O(a^2)$.

B_c = rectangular box approx. S_c of same width.

BCZ cocycle

Let $\theta^{(n)} : \Omega \to \mathbb{R}$ be defined by

$$\theta^{(n)}(x,y) = \frac{1}{xy} - \frac{n^2}{A} \qquad \left(\frac{n^2}{A} \to \frac{\pi^2}{3}\right) \text{ as } n \to \infty$$

and

$$\chi_j^{(n)}(x,y) = \sum_{1}^{j} \theta^{(n)}(T^{i-1}(x,y)) \qquad \text{“additive cocycle”}$$

Then

$$\boxed{RH \iff \frac{1}{A}\sum_{1}^{A} \left|\chi_j^{(n)}(x_n,1)\right| = O(n^{\frac{1}{2}+\varepsilon})}$$

$$\iff \frac{1}{A}\sum_{1}^{A} \left|\,-\,''\,-\,\right|^2 = O(n^{1+\varepsilon})$$

Remark: For a.e. $\omega \in \Omega$, $\left|\chi_n^{(\infty)}(\omega)\right| = O(n^{\frac{1}{2}+\varepsilon})$

Q: What other interesting number theory questions can be reformulated as questions about BCZ map?