Linear independence of values of G-functions

Moscow seminar in Diophantine analysis

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Joint work with Tanguy RIVOAL.

I Setting

Def (Siegel, 1929): a G-function is a power series $F(z) = \sum_{k=0}^{\infty} a_k z^k \in \overline{\mathbb{Q}}[[z]]$ such that:

(i) $\exists c > 0 \ \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ \forall k \in \mathbb{N} \quad |\sigma(a_k)| \leq c^{k+1}$.

(ii) Let $D_m \in \mathbb{N}^*$ be minimal such that $D_m a_0, \ldots, D_m a_m$ are alg. integers. Then $\exists D > 0 \ \forall m \in \mathbb{N} \quad D_m \leq D^{m+1}$.

(iii) $F(z)$ is solution of a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

Examples:
- $F(z) \in \overline{\mathbb{Q}}[z]$; $F(z) = -\log(1-z) = \mathrm{Li}_1(z)$; $F(z) = \mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$, $s \geq 1$: s-th polylogarithm
- Algebraic functions holomorphic at 0
- $F(z) = {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} \middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{p+1})_k}{(1)_k (b_1)_k \cdots (b_p)_k} z^k$ hypergeometric series with $a_i, b_j \in \mathbb{Q}$, $-b_j \notin \mathbb{N} = \{0, 1, 2, \ldots\}$

  Pochhammer symbol: $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$

Unexpected algebraic values: ${}_2F_1\left[\begin{matrix} \frac{1}{12}, \frac{5}{12} \\ 1/2 \end{matrix} \middle| \frac{3^3 \times 7^2}{11^3}\right] = \frac{3}{4}\sqrt[4]{11}$ (Beukers-Wolfart, 1986)

$\rightarrow$ No general conjecture to predict if $F(z)$ is algebraic or transcendental.

Classical results (Siegel, Galochkin, Bombieri, Chudnovsky, André, ...):

* Fix G-functions $F_1, \dots, F_s$.
* Let $z \in \overline{\mathbb{Q}}^*$ be sufficiently close to 0 (in terms of $F_1, \dots, F_s$, $[\mathbb{Q}(z):\mathbb{Q}]$, ...)
* Prove Diophantine results on the $F_j(z)$.

Th (Nikishin, 1979): Let $p, q \in \mathbb{N}^*$ be such that $q > p^{s+1}(4es)^{s(s-1)} 2^{s-1}$ with $s \geq 1$. Then $1, \mathrm{Li}_1(\frac{-p}{q}), \mathrm{Li}_2(\frac{-p}{q}), \dots, \mathrm{Li}_s(\frac{-p}{q})$ are linearly independent over $\mathbb{Q}$.

Alternative point of view (Rivoal, Ball-Rivoal, ...):

* Fix a point $z \in \overline{\mathbb{Q}}^*$
* Consider many G-functions $F_1, \dots, F_s$
* Bound from below $\dim_{\mathbb{K}} \mathrm{Span}_{\mathbb{K}}(F_1(z), \dots, F_s(z))$ where $\mathbb{K}$ is a fixed number field.

With $\mathrm{Li}_s(1) = \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$:

Th (Ball-Rivoal, 2000): $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \dots, \zeta(s)) \geq \frac{1+o(1)}{1+\log 2} \log s$ as $s \longrightarrow +\infty$ (with s odd)

Corollary: $\zeta(s) \notin \mathbb{Q}$ for infinitely many odd integers $s \geq 3$.

Apéry, 1978: $\zeta(3) \notin \mathbb{Q}$.

For $s \geq 2$ even: $\zeta(s)\pi^{-s} \in \mathbb{Q}^*$ so that $\zeta(s)$ is transcendental.

Conjecture: $\zeta(s)$ is transcendental for any $s \geq 2$.

There is no odd $s \geq 5$ for which $\zeta(s)$ is known to be irrational.
[Zudilin, F.-Sprang-Zudilin, Lai-Yu, ...]

<u>Th</u> (Rivoal, 2003) : For $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$, $0<|\alpha|<1$: $\dim_{\mathbb{Q}(\alpha)} \mathrm{Span}_{\mathbb{Q}(\alpha)} (1, \mathrm{Li}_1(\alpha), \dots, \mathrm{Li}_s(\alpha)) \geq \dfrac{(1+o(1)) \log s}{(1+\log 2)\,[\mathbb{Q}(\alpha):\mathbb{Q}]}$ as $s \to +\infty$.

<u>Th</u> (Marcovecchio, 2006) : also for $\alpha \in \overline{\mathbb{Q}}$ non-real, $0 < |\alpha| < 1$.

<u>Th 1</u> : The same holds for any $\alpha \in \overline{\mathbb{Q}}^* \setminus [1, +\infty)$ using analytic continuation of polylogarithms [except that $1+\log 2$ is replaced by a computable absolute constant]

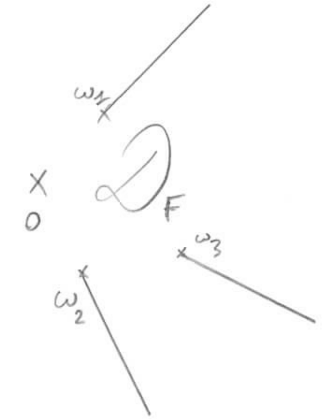
II Notation and results.

$F(z) = \sum_{k=0}^{\infty} a_k z^k$ a non-polynomial G-function, $m, s \geq 1$; set :

$$\boxed{F_m^{[s]}(z) = \sum_{k=0}^{\infty} \frac{a_k}{(k+m)^s} z^{k+m}} \quad \text{(G-function)}$$

<u>Example</u> : $F(z) = \frac{1}{1-z} \rightsquigarrow \forall k \in \mathbb{N}$ $a_k = 1$ and $F_m^{[s]}(z) = \mathrm{Li}_s(z) - \sum_{j=0}^{m-1} \frac{z^j}{j^s}$.

Fix
- $\mathbb{K}$ number field containing all a_k
- $\mathcal{D}_F$ domain, star-shaped at 0, on which F is analytic: $\mathcal{D}_F = \mathbb{C} \setminus \bigcup_{\omega \text{ finite singularity of } F} \{\omega t, t \in \mathbb{R}, t \geq 1\}$
- $L \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$, $L \neq 0$, of minimal order such that $LF = 0$.



Let $\alpha \in \mathbb{K}^* \cap \mathcal{D}_F$ such that α is not a singularity of L.

For $s \geq 1$ let :

$$\boxed{\Phi_s = \mathrm{Span}_{\mathbb{K}} \{ F_m^{[\sigma]}(\alpha),\ m \geq 1,\ 1 \leq \sigma \leq s \}}$$

<u>Th 2</u> : $\dim_{\mathbb{K}} \Phi_s \leq \ell s + \mu$ where ℓ and μ depend only on F.

<u>Idea</u> : Φ_s is generated by the $F_m^{[\sigma]}(\alpha)$, $1 \leq m \leq \ell$, $1 \leq \sigma \leq s$, and at most μ other numbers.

<u>Th 3</u> : $\dim_{\mathbb{K}} \Phi_s \geq \dfrac{\log s}{C \cdot [\mathbb{K}:\mathbb{Q}]}$ for any s suff.^{ly} large, where $C>0$ can be computed in terms of F only.

<u>Special case</u> : $F(z) = \frac{1}{1-z}$, $\mathcal{D}_F = \mathbb{C} \setminus [1, +\infty)$, $\mathbb{K} = \mathbb{Q}(\alpha)$, $\Phi_s = \mathrm{Span}_{\mathbb{Q}(\alpha)} (1, \mathrm{Li}_1(\alpha), \dots, \mathrm{Li}_s(\alpha)) \rightsquigarrow$ Th 1.

Corollary: Let $p \geq 0$, $a_1, \dots, a_{p+1}, b_1, \dots, b_p \in \mathbb{Q}$ with $a_i \notin \mathbb{Z} \setminus \{1\}$ and $-b_j \notin \mathbb{N}$. Let $\alpha \in \overline{\mathbb{Q}}^* \setminus [1, +\infty)$.

Then $\dim_{\mathbb{Q}(\alpha)} \mathrm{Span}_{\mathbb{Q}(\alpha)} \Big\{ \underbrace{{}_{p+\sigma+1}F_{p+\sigma}\left[\begin{matrix} a_1, \dots, a_{p+1}, 1, \dots, 1 \\ b_1, \dots, b_p, \underbrace{2, \dots, 2}_{\sigma} \end{matrix} \middle| \alpha \right]}, \ 1 \leq \sigma \leq s \Big\} \geq \dfrac{\log s}{c \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}]}$ for any s suff${}^{\text{ly}}$ large, with $c = c(a_i, b_j)$.

$\hookrightarrow$ if $|\alpha| < 1$ this is $\displaystyle\sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{p+1})_k}{k!(b_1)_k \cdots (b_p)_k} \frac{\alpha^k}{(k+1)^\sigma}$

In particular, not all these numbers are zero!

III Upper bound.

> **Recall**: $F(z) = \sum_{k=0}^{\infty} a_k z^k$ G-function $\qquad F_n^{[s]}(z) = \sum_{k=0}^{\infty} \dfrac{a_k}{(k+n)^s} z^{k+n}$
>
> $L \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$, $L \neq 0$, of minimal order such that $LF = 0$.

Let μ denote the order of L ($=$ degree in $\frac{d}{dz}$).

Prop 1: There exist $\ell \in \mathbb{N}$, $\kappa_{j,t,s,n} \in \mathbb{K}$, $P_{j,s,n} \in \mathbb{K}[z]$ such that for any $n, s \geq 1$ and any $z \in \mathcal{D}_F$:

$$F_n^{[s]}(z) = \sum_{t=1}^{s} \sum_{j=1}^{\ell} \kappa_{j,t,s,n} F_j^{[t]}(z) + \sum_{j=0}^{\mu-1} P_{j,s,n}(z) \left(\left(z \frac{d}{dz}\right)^j F \right)(z).$$

Moreover: as $n \to \infty$, geometric growth of denominators & moduli of Galois conjugates of $\kappa_{j,t,s,n}$ and coefficients of $P_{j,s,n}(z)$.

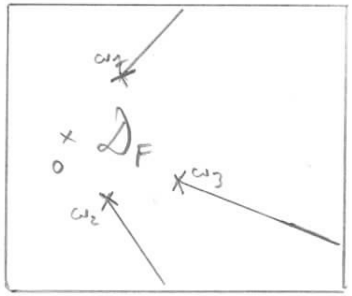
$\rightsquigarrow \Phi_s = \mathrm{Span}_{\mathbb{K}} \{ F_n^{[\sigma]}(\alpha),\ n \geq 1,\ 1 \leq \sigma \leq s \}$ is spanned by the $F_j^{[t]}(\alpha)$, $1 \leq j \leq \ell$, $1 \leq t \leq s$ and the $\left(\left(z \frac{d}{dz}\right)^j F\right)(\alpha)$, $0 \leq j \leq \mu - 1$

$\rightsquigarrow$ Th 2: $\dim_{\mathbb{K}} \Phi_s \leq \ell s + \mu$.

$\boxed{5}$

IV Lower bound.

$$\boxed{\begin{array}{ll} F(z) = \sum_{k=0}^{\infty} a_k z^k \ \text{ G-function} & F_m^{[s]}(z) = \sum_{k=0}^{\infty} \frac{a_k}{(k+m)^s} z^{k+m} \\ \mathbb{K} \text{ number field, } \forall k \ a_k \in \mathbb{K} & L \text{ such that } LF = 0 \end{array}}$$



Fix $\alpha \in \mathbb{K}^* \cap \mathcal{D}_F$
such that α is not a singularity of L.
Choose S sufficiently large, and let $r = \left\lfloor \frac{S}{(\log S)^2} \right\rfloor$.
For $|z| < R$ (= radius of CV of F) and $m \geq 0$:

$$\boxed{J_m(z) = m!^{S-r} \sum_{k=0}^{\infty} \frac{(k - rm + 1)_{rm}}{(k+1)_{m+1}^S} \underbrace{a_k}_{\text{Taylor coeff. of } F(z)} z^{k+m+1}}$$

If $a_k = 1$: construction used by Rivoal and Marcovecchio

Integral representation:
$$\boxed{J_m(z) = \frac{z^{(r+1)m+1}}{m!^r} \int_{[0,1]^S} F^{(rm)}(z t_1 \cdots t_S) \prod_{j=1}^{S} t_j^{rm} (1 - t_j)^m \, dt_1 \ldots dt_S}$$

$\rightsquigarrow$ analytic continuation of $J_m(z)$ to $\mathcal{D}_F$ (since $z \in \mathcal{D}_F \Rightarrow \forall t_1, \ldots, t_S \in [0,1] \ \ z t_1 \cdots t_S \in \mathcal{D}_F$)

Prop 2: $\limsup_{m \to +\infty} |J_m(\alpha)|^{1/m} \leq \frac{c(\alpha)^r}{r^{S-r}}$ where $c(\alpha)$ depends only on α and F (not on S, r).

⚠ Surprise (to me): $|J_m(\alpha)|$ is small even when α is outside of the disk of convergence.
↳ essentially as $\frac{1}{r^S}$ since $r = \left\lfloor \frac{S}{(\log S)^2} \right\rfloor$

Proof: the important part is $\int_{[0,1]^S} \prod_{j=1}^{S} t_j^{rm} (1 - t_j)^m \, dt_1 \ldots dt_S = \left(\frac{(rm)! \, m!}{((r+1)m + 1)!} \right)^S \approx \left(\frac{r^r}{(r+1)^{r+1}} \right)^{Sm} \approx \left(\frac{1}{re} \right)^{Sm}$.

$$F(z) = \sum_{k=0}^{\infty} a_k z^k \qquad F_m^{[s]}(z) = \sum_{k=0}^{\infty} \frac{a_k}{(k+m)^s} z^{k+m}$$

$$J_m(z) = m!^{S-r} \sum_{k=0}^{\infty} \frac{(k-rm+1)_{rm}}{(k+1)_{m+1}^S} a_k z^{k+m+1}$$

<u>Expansion of $J_m(\alpha)$ as linear form in the $F_j^{[t]}(\alpha)$ and $\left(\left(z\frac{d}{dz}\right)^j F\right)(\alpha)$:</u>

<u>Recall</u>: $\left[\begin{array}{l}\alpha \in \mathbb{K}^* \cap \mathcal{D}_F \text{ not a singularity of } L \\ S \text{ large enough}, \quad r = \left\lfloor \frac{S}{(\log S)^2} \right\rfloor\end{array}\right.$

Denote by $\theta_1, \dots, \theta_N$ the numbers $\begin{cases} F_j^{[t]}(\alpha), & 1 \leq j \leq \ell,\ 1 \leq t \leq S \\ \left(\left(z\frac{d}{dz}\right)^j F\right)(\alpha), & 0 \leq j \leq \mu - 1 \end{cases}$
$(N = \ell S + \mu)$

Prop 1 $\rightsquigarrow$ for any $j \geq 1$ and $\sigma \in [\![1, S]\!]$, $F_j^{[\sigma]}(\alpha) \in \mathrm{Span}_{\mathbb{K}}(\theta_1, \dots, \theta_N) = \Phi_S$.

Partial fraction expansion: $\dfrac{(k-rm+1)_{rm}}{(k+1)_{m+1}^S} = \displaystyle\sum_{j=1}^{m+1} \sum_{\sigma=1}^{S} \frac{c_{j,\sigma}}{(k+j)^\sigma}$. Plugged into the def of $J_m(\alpha)$, yields:

<u>Prop 3</u>: There exists a sequence (δ_m) of positive integers such that:

(i) $\delta_m J_m(\alpha) = p_{1,m}\theta_1 + \dots + p_{N,m}\theta_N$ with $p_{j,m} \in \mathcal{O}_{\mathbb{K}}$ (algebraic integers in $\mathbb{K}$)

(ii) $\forall j \in [\![1, N]\!]$ $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ $\limsup_{m \to +\infty} |\sigma(p_{j,m})|^{1/m} \leq b_0$ with an explicit $b_0(\alpha, F, S, r) > 1$.

(iii) $\limsup_{m \to +\infty} |\delta_m J_m(\alpha)|^{1/m} \leq a_0$ with an explicit $a_0(\alpha, F, S, r) < 1$.

<u>Values of a_0 and b_0</u>: $\quad a_0 \leq \dfrac{c_1^S}{r^S}$ and $b_0 \leq c_2^S$ where c_1 and c_2 depend only on F and α.

Just one assumption is missing to apply a linear independence criterion and deduce that:

$$\dim_{\mathbb{K}} \Phi_S = \dim_{\mathbb{K}} \mathrm{Span}_{\mathbb{K}}(\theta_1, \dots, \theta_N) \geq \frac{1}{[\mathbb{K}:\mathbb{Q}]}\left(1 - \frac{\log a_0}{\log b_0}\right) \underset{\substack{\text{as } S \to +\infty \\ r = \left\lfloor \frac{S}{(\log S)^2} \right\rfloor}}{\sim} \frac{\log S}{[\mathbb{K}:\mathbb{Q}] \log(c_2)}$$
and Th 3 follows.

V Inside the CV disk: Nesterenko's linear independence criterion.

Assume that $|\alpha| < R$ = radius of CV of $F(z) = \sum_{k=0}^{\infty} a_k z^k$.

Saddle point method: $\delta_m J_m(\alpha) = a_0^{m(1+o(1))} \left(\sum_{q=1}^{Q} c_q \zeta_q^m + o(1) \right)$ with $\begin{cases} c_1, \dots, c_Q \in \mathbb{C}^* \\ \zeta_1, \dots, \zeta_Q \in \mathbb{C} \text{ pairwise distinct with } |\zeta_q| = 1 \end{cases}$

<u>Th 4</u> (version of Nesterenko's linear indep. criterion): Let $\theta_1, \dots, \theta_N \in \mathbb{C}$ and $\mathbb{K}$ be a number field; if $\mathbb{K} \subset \mathbb{R}$ assume $\theta_1, \dots, \theta_N \in \mathbb{R}$.

Let $\ell_{j,m} \in \mathcal{O}_{\mathbb{K}}$ be such that
$$\begin{cases} \forall j \in [\![1, N]\!] \quad \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \quad \limsup_{m \to +\infty} |\sigma(\ell_{j,m})|^{1/m} \leq b \quad \text{for some } b > 1 \\ \ell_{1,m}\theta_1 + \dots + \ell_{N,m}\theta_N = a^{m(1+o(1))} \left(\sum_{q=1}^{Q} c_q \zeta_q^m + o(1) \right) \text{ with } c_q, \zeta_q \text{ as above and } 0 < a < 1 \end{cases}$$

Then $\dim_{\mathbb{K}} \mathrm{Span}_{\mathbb{K}}(\theta_1, \dots, \theta_N) \geq \frac{1}{[\mathbb{K} : \mathbb{Q}]} \left(1 - \frac{\log a}{\log b} \right)$.

Nesterenko, 1985: $\mathbb{K} = \mathbb{Q}$ and $|\ell_{1,m}\theta_1 + \dots + \ell_{N,m}\theta_N| = a^{m(1+o(1))}$ [important point: not too small]

Töpfer (1994), Bedulev (1998): any $\mathbb{K}$, same asymptotic behavior.

Sorokin (2007), F. (2011): $\mathbb{K} = \mathbb{Q}$, $|\ell_{1,m}\theta_1 + \dots + \ell_{N,m}\theta_N| = a^{m(1+o(1))} \left(|\cos(m\omega + \varphi)| + o(1) \right)$ $\begin{bmatrix} \text{Sorokin: linear forms in log. } [\text{if } e^{i\omega}, e^{i\varphi} \in \overline{\mathbb{Q}}] \\ \text{F.: Kronecker - Weyl's equidistribution theorem} \end{bmatrix}$

Proof of Th 4: Among Q successive values of m, there is always at least one for which $\sum_{q=1}^{Q} c_q \zeta_q^m$ is not too small, otherwise the Vandermonde determinant $\left| [\zeta_q^m]_{\substack{1 \leq q \leq Q \\ m_0 \leq m \leq m_0 + Q - 1}} \right|$ would be too small in modulus $\rightsquigarrow$ Apply Töpfer's theorem to this subsequence.

VI Outside the CV disk: Shidlovsky's lemma and Siegel's linear indep. criterion.

Th (version of Siegel's linear indep. criterion): Let $\theta_1, \dots, \theta_N \in \mathbb{C}$ and $\mathbb{K}$ be a number field.

Let $p_{j,m}^{(i)} \in \mathcal{O}_{\mathbb{K}}$ be such that
$$\begin{cases} \forall i,j \in [\![1,N]\!] \quad \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \quad \limsup_{m \to +\infty} \left|\sigma(p_{j,m}^{(i)})\right|^{1/m} \leq b \quad \text{for some } b > 1 \\ \forall i \in [\![1,N]\!] \quad \left|p_{1,m}^{(i)}\theta_1 + \dots + p_{N,m}^{(i)}\theta_N\right| \leq a^{m(1+o(1))} \quad \text{for some } a < 1 \\ \text{For any } m \text{ sufficiently large the matrix } \left[p_{j,m}^{(i)}\right]_{1 \leq i,j \leq N} \text{ is invertible} \end{cases}$$

Then $\dim_{\mathbb{K}} \mathrm{Span}_{\mathbb{K}}(\theta_1, \dots, \theta_N) \geq \frac{1}{[\mathbb{K}:\mathbb{Q}]}\left(1 - \frac{\log a}{\log b}\right)$.

$\rightsquigarrow$ Instead of just $J_m(\alpha)$, construct N linear forms in $\theta_1, \dots, \theta_N$: $J_m^{(\kappa)}(\alpha)$ for some κ.

$$\boxed{J_m^{(\kappa)}(z) = \sum_{j=1}^{\ell} \sum_{t=1}^{s} P_{j,t,m,\kappa}(z)\, F_j^{[t]}(z) + \sum_{j=0}^{\mu-1} \widetilde{P}_{j,m,\kappa}(z) \left(\left(z\frac{d}{dz}\right)^j F\right)(z) \quad \text{with } P_{j,t,m,\kappa}(z),\ \widetilde{P}_{j,m,\kappa}(z) \in \mathbb{K}(z) \text{ (polynomials if } \kappa = 0)}$$

Step 1: Prove that the polynomials $P_{j,t,m,0}(z)$ and $\widetilde{P}_{j,m,0}(z)$ are solution of a Padé approximation problem with as many equations as the number of unknowns (up to an additive constant) indep. of m

(Involves other solutions of $Ly = 0$; uses results of André, Chudnovsky, Katz).

Step 2: Prove a general version of Shidlovsky's lemma.
(Follows the approach of Bertrand-Beukers (1985), Bertrand (2012), F. (2018) based on differential Galois theory)

Step 1 + Step 2 + Siegel's criterion $\rightsquigarrow$ Th 3.

Recall that $J_m(z) = m!^{S-r} \sum_{k=0}^{\infty} \frac{(k-rm+1)_{rm}}{(k+1)_{m+1}^{S}} a_k z^{k+m+1}$

For any solution f of $Lf=0$ consider

$$J_{f,m}(z) = \sum_{j=1}^{\ell} \sum_{t=1}^{S} P_{j,t,m,0}(z) \underbrace{f_j^{[t]}(z)} + \sum_{j=0}^{\mu-1} \tilde{P}_{j,m,0}(z) \left(\left(z \frac{d}{dz}\right)^j f\right)(z)$$

$$\longrightarrow \begin{cases} f_j^{[1]}(z) = \int^z x^{j-1} f(x)\,dx \\ f_j^{[t+1]}(z) = \int^z \frac{1}{x} f_j^{[t]}(x)\,dx \end{cases}$$

($\triangle$ choice of the constant of integration)

Padé-approximation problem:

as $z \to 0$, $J_{f,m}(z) = \begin{cases} O\left(z^{(r+1)m+1}\right) & \text{for any } f \text{ holomorphic at } 0 \\ O\left(z^{m-m_0}\right) & \text{for all other solutions of } Lf=0 \end{cases}$ (with m_0 independent from m) $(m_0 = m_0(L, r, S))$

& conditions on $J_{f,m}(z)$ at all singularities of L

& another condition at 0.