MULTIPLICATIVELY BADLY APPROXIMABLE MATRICES AND THE LITTLEWOOD CONJECTURE

The Littlewood conjecture states that for all vectors $(\alpha, \beta) \in \mathbb{R}^2$ we have

$$\liminf_{q \to +\infty} q\|q\alpha\|\|q\beta\| = 0, \tag{1}$$

where $\| \cdot \|$ denotes the distance from the nearest integer. As of today, this conjecture remains open, event though we know that the set of its counterexamples has zero Hausdorff dimension.

Several weaker statements have been considered, such as

$$\liminf_{q \to +\infty} q \log(q)^{\lambda} \|q\alpha\|\|q\beta\| = 0, \tag{2}$$

for all $(\alpha, \beta) \in \mathbb{R}^2$, where $\lambda > 0$. Can we say for which values of $\lambda > 0$ the set of counterexamples to (2) is non-empty? A trivial application of Gallagher's Theorem shows that if $\lambda > 2$ Equation (2) holds for almost all vectors $(\alpha, \beta) \in \mathbb{R}^2$, whereas for $\lambda \leq 2$ it almost never holds. Moshchevitin and Bugeaud showed that for $\lambda = 2$, the set of counterexamples to (2) has full Hausdorff dimension and later Badziahin showed the same for $\lambda > 1$.

Surprisingly, *nothing* is known when $\lambda \leq 1$. In particular, for $\lambda \leq 1$ Statement (2) could be true, which would be a significant strengthening of the Littlewood conjecture. In this talk we discuss this problem and look at its multi-dimensional generalisations.