Exceptional algebraic values of E-functions

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Definition of E-functions

We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1 (adapted from Siegel, 1929)

An E-function is a power series $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$

(i) *$F(z)$ is solution of an homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

and there exists $C > 0$ s.t.

(ii) *For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.*

(iii) *For any $n \geq 0$, there exists $d_n \in \mathbb{N}$ s.t. $0 < |d_n| \leq C^{n+1}$ and $d_n a_m \in \mathcal{O}_{\overline{\mathbb{Q}}}$ for all $0 \leq m \leq n$.*

Siegel: Eine Funktion y, deren Potenzreihe diese drei Eigenschaften hat, möge kurz eine E-Funktion genannt werden. Offenbar ist die Exponentialfunktion eine E-Funktion.

A function y whose power series has these three properties shall be called an E-function. The exponential function is obviously an E-function.

Siegel's original definition was in fact slightly more general, but both definitions are now believed to be equivalent.

Examples

Polynomials in $\overline{\mathbb{Q}}[z]$,

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{4^n} \frac{z^{2n}}{(2n)!},$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} \Big) z^n = e^{3z} J_0(2i\sqrt{2}z),$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right) z^n$$

Non-polynomial algebraic functions, $-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ and $J_0(\sqrt{z})$ are not E-functions.

Structural properties of E-functions

E-functions are entire functions; they form a ring, stable by $\frac{d}{dz}$ and $\int_0^z$.

Its units are of the form $\alpha e^{\beta z}$, where $\alpha \in \overline{\mathbb{Q}}^*$ and $\beta \in \overline{\mathbb{Q}}$ (André 2000).

For $q \geq p \geq 1$, the generalized hypergeometric function

$$ {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z^{q-p}\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^{(q-p)n}}{n!}, $$

is an E-function if the a_j's and b_j's are in $\mathbb{Q}$. The converse is false: take $q = p = 1$ and $a_1 = \sqrt{2} + 1$, $b_1 = \sqrt{2}$. Necessary and sufficient conditions have been given by Galochkin (1981).

Siegel's problem: Is any E-function a finite $\overline{\mathbb{Q}}$-linear combination of finite products of ${}_pF_q(z^{q-p})$ series?

The answer is yes if the E-function satisfies a linear differential equation of order ≤ 2 (Gorelov, 2004). The answer is no in general for E-functions of differential order 3 (Fresán-Jossen, 2020).

Diophantine properties of the exponential function

Siegel defined E-functions to generalize the Diophantine properties of exp.

Theorem 1 (Hermite-Lindemann)

For any $\alpha \in \overline{\mathbb{Q}}^*$, $e^\alpha \notin \overline{\mathbb{Q}}$.

More generally,

Theorem 2 (Lindemann-Weierstrass)

Let $\alpha_1, \ldots, \alpha_k \in \overline{\mathbb{Q}}$ be $\mathbb{Q}$-linearly independent. Then $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are $\overline{\mathbb{Q}}$-algebraically independent.

Equivalently:

Let $\alpha_1, \ldots, \alpha_k \in \overline{\mathbb{Q}}$ be pairwise distinct. Then $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are $\overline{\mathbb{Q}}$-linearly independent.

The Siegel-Shidlovskii Theorem

$Y(z) = {}^t(F_1(z), \ldots, F_n(z))$ a vector of E-functions solution of a differential system $Y'(z) = M(z)Y(z)$ where $M(z) \in M_n(\overline{\mathbb{Q}}(z))$.

$T(z) \in \overline{\mathbb{Q}}[z]$ the least common denominator of the entries of $M(z)$.

Theorem 3 (Siegel-Shidlovskii 1929, 1956)

For any $\alpha \in \overline{\mathbb{Q}}$ s.t. $\alpha T(\alpha) \neq 0$,

$$\mathrm{degtr}_{\overline{\mathbb{Q}}(z)}\big(F_1(z), \ldots, F_n(z)\big) = \mathrm{degtr}_{\overline{\mathbb{Q}}}\big(F_1(\alpha), \ldots, F_n(\alpha)\big).$$

Problem 1: If $\mathrm{degtr}_{\overline{\mathbb{Q}}(z)}(F_1(z), \ldots, F_n(z)) < n$, the theorem does not imply that $F_1(\alpha) \notin \overline{\mathbb{Q}}$ at any $\alpha \in \overline{\mathbb{Q}}$ s.t. $\alpha T(\alpha) \neq 0$.

Problem 2: It does not say anything about the Diophantine nature of $F_1(\alpha)$ when $T(\alpha) = 0$ and $\alpha \neq 0$.

Beyond Siegel and Shidlovskii

Beukers refined a theorem of Nesterenko and Shidlovskii (1996).

Theorem 4 (Beukers 2006)

In the same setting as before, consider $\alpha \in \overline{\mathbb{Q}}$ *s.t.* $\alpha T(\alpha) \neq 0$. *Assume that* $P\big(F_1(\alpha), \ldots, F_n(\alpha)\big) = 0$ *for some* $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ *homogeneous. Then, there exists* $Q \in \overline{\mathbb{Q}}[Z, X_1, \ldots, X_n]$ *homogeneous in the* X_j*'s s.t.*

$$Q\big(z, F_1(z), \ldots, F_n(z)\big) = 0 \quad \text{and} \quad Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).$$

Theorem 5 (Beukers' Corollary 1.4)

Assume that $F_1(z), \ldots, F_n(z)$ *are* $\overline{\mathbb{Q}}(z)$*-linearly independent. Then for any* $\alpha \in \overline{\mathbb{Q}}$ *s.t.* $\alpha T(\alpha) \neq 0$, *the number* $F_1(\alpha), \ldots, F_n(\alpha)$ *are* $\overline{\mathbb{Q}}$*-linearly independent.*

When $T(\alpha) = 0$, the relation

$$0 = \lim_{z \to \alpha} T(z) Y'(z) = \lim_{z \to \alpha} T(z) M(z) Y(z)$$

implies that $F_1(\alpha), \ldots, F_n(\alpha)$ are $\overline{\mathbb{Q}}$-linearly dependent.

Exceptional algebraic values of E-functions

Theorem 6 (Adamczewski-R., 2017)

There exists an algorithm to perform the following tasks.

Given an E-function $f(z)$ as input, it first says whether $f(z)$ is transcendental or not.

If it is transcendental, it then outputs the finite list of algebraic numbers α such that $f(\alpha)$ is algebraic, together with the corresponding list of values $f(\alpha)$.

How is the input given?

• Let $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ be an E-function. By definition, $Lf(z) = 0$ for some $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ or equivalently $Ra_n = 0$ for some $R \in \overline{\mathbb{Q}}(n)[Shift]$.

The expression "Given an E-function $f(z)$" means that

(i) One knows explicitly $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ s.t. $Lf(z) = 0$.

(ii) One knows enough Taylor coefficients of $f(z)$ to be able to compute from L as many coefficients as needed.

In general, no explicit formulas are known for the solutions of a given $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$.

No algorithm is known to decide if L has an E-function as solution.

(iii) An oracle guarantees that $f(z)$ is an E-function.

• In practice, an E-function is given by an explicit expression for its Taylor coefficients as a multiple hypergeometric sum.

Both L and R can then be computed in principle using algorithms à la Zeilberger.

1^{st} step: minimal equation

Input: f and L, of order r_0 and degree δ_0.

Output: $L_{\min} \in \overline{\mathbb{Q}}[z, \frac{d}{dz}] \setminus \{0\}$ such that $L_{\min} f(z) = 0$ and minimal for the order.

• Grigoriev (1991), Bostan-Rivoal-Salvy (2019) : there exist an explicit δ_1 and $L_{\min}$ s.t. $\deg(L_{\min}) \leq \delta_1$. Obviously, $\mathrm{ord}(L_{\min}) \leq r_0$.

• Let $1 \leq r \leq r_0$ and $0 \leq \delta \leq \delta_1$. For any $P_0(z), \ldots, P_r(z) \in \overline{\mathbb{Q}}[z]$ not all zero, of degrees $\leq \delta$, set

$$R(z) := P_0(z) f(z) + \cdots + P_r(z) f^{(r)}(z).$$

Bertrand-Beukers (1985), Bertrand-Chirskii-Yebbou (2004): There exists an explicit integer N s.t.

$$R \equiv 0 \iff \mathrm{ord}_{z=0} R(z) \geq N.$$

• Deciding if $R \equiv 0$ amounts to finding a non-trivial element in the kernel of an $(r+1)(\delta+1) \times (N+1)$ matrix with algebraic entries that depend on the first $N+1$ Taylor coefficients of f.

$L_{\min}$ will eventually be found.

2^{nd} step: minimal inhomogeneous equation

Input: f and $L_{\min}$ written in the form

$$\sum_{j=0}^{r} P_j(z) f^{(j)}(z) = 0, \qquad P_j(z) \in \overline{\mathbb{Q}}(z) \text{ and } P_r(z) \equiv 1\,.$$

Output: A minimal non-zero inhomogenous equation $L_{inhom} f(z) = 0$ of order s, with coefficients in $\overline{\mathbb{Q}}(z)$.

- Necessarily, $s \in \{r, r-1\}$.
- If $s = r-1$, write L_{inhom} in the form

$$1 + \sum_{j=0}^{r-1} Q_j(z) f^{(j)}(z) = 0, \qquad Q_j(z) \in \overline{\mathbb{Q}}(z).$$

The Q_j's are solutions of the system

$$\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_{r-1} \end{pmatrix}' = \begin{pmatrix} 0 & 0 & \dots & 0 & P_0 \\ -1 & 0 & \dots & 0 & P_1 \\ 0 & -1 & \dots & 0 & P_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & P_{r-1} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_{r-1} \end{pmatrix}. \tag{1}$$

• There exist algorithms to decide whether a given differential system with coefficients in $\overline{\mathbb{Q}}(z)$ has a non-zero vector of rational solutions (and then compute them) or not.

• If (1) has no such rational vector, then $s = r$ and we set $L_{inhom} := L_{\min}$.

• If (1) has a non-zero vector of rational solutions A_j's, then by construction of (1),

$$\sum_{j=0}^{r-1} A_j(z) f^{(j)}(z) = c \tag{2}$$

for some $c \in \overline{\mathbb{Q}}$ to be determined.

The A_j's are explicitely known and we know as many Taylor coefficients of f as neededat $z = 0$, the constant c can be explicitely computed.

The resulting explicit equation (2) is L_{inhom}.

3^{rd} step: capturing the exceptional algebraic values of f

Input: f and L_{inhom} of order s.

- If $s = 0$, then f is a polynomial and the algorithm stops here.
- If $s \geq 1$, then f is transcendental over $\mathbb{C}(z)$. Rewrite L_{inhom} as

$$\begin{pmatrix} 0 \\ f'(z) \\ f''(z) \\ \vdots \\ f^{(s)}(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{u_1(z)}{u_0(z)} & \frac{u_2(z)}{u_0(z)} & \cdots & \cdots & \cdots & \frac{u_{s+1}(z)}{u_0(z)} \end{pmatrix} \begin{pmatrix} 1 \\ f(z) \\ f'(z) \\ \vdots \\ f^{(s-1)}(z) \end{pmatrix} \tag{3}$$

where the u_j's are in $\overline{\mathbb{Q}}[z]$, with $u_0 \not\equiv 0$.

- The functions $1, f(z), \ldots, f^{(s-1)}(z)$ are $\overline{\mathbb{Q}}(z)$-linearly independent.

By Corollary 1.4, when $\alpha \in \overline{\mathbb{Q}}$ and $\alpha u_0(\alpha) \neq 0$, the numbers $1, f(\alpha), \ldots, f^{(s-1)}(\alpha)$ are $\overline{\mathbb{Q}}$-linearly independent. In particular, $f(\alpha) \notin \overline{\mathbb{Q}}$.

- In other words, if $\alpha \in \overline{\mathbb{Q}}$ and $f(\alpha) \in \overline{\mathbb{Q}}$, then $\boxed{\alpha u_0(\alpha) = 0}$.

Last steps

Goal: Given $\alpha \neq 0$ such that $u_0(\alpha) = 0$, decide whether $f(\alpha) \in \overline{\mathbb{Q}}$ or not.

• Beukers' Theorem 1.5: there exists an $(s+1) \times (s+1)$ invertible matrix $\mathcal{M}(z)$ with entries in $\overline{\mathbb{Q}}[z]$ such that

$$\begin{pmatrix} 1 \\ f(z) \\ \vdots \\ f^{(s-1)}(z) \end{pmatrix} = \mathcal{M}(z) \begin{pmatrix} e_0(z) \\ e_1(z) \\ \vdots \\ e_s(z) \end{pmatrix},$$

where the e_j's are E-functions solutions of a differential system with entries in $\overline{\mathbb{Q}}[z, 1/z]$.

• The e_j's are $\overline{\mathbb{Q}}(z)$-linearly independent. By Corollary 1.4, when $\alpha \in \overline{\mathbb{Q}}^*$, the numbers

$$e_1(\alpha), e_2(\alpha), \ldots, e_s(\alpha)$$

are $\overline{\mathbb{Q}}$-linearly independent.

• $f(\alpha) \in \overline{\mathbb{Q}}$ if and only if there exists $\lambda = (\beta, 1, 0, \ldots, 0) \in \overline{\mathbb{Q}}^{s+1}$ s.t.

$$0 = \lambda \cdot \begin{pmatrix} 1 \\ f(\alpha) \\ \vdots \\ f^{(s-1)}(\alpha) \end{pmatrix} = \lambda \mathcal{M}(\alpha) \begin{pmatrix} e_0(\alpha) \\ e_1(\alpha) \\ \vdots \\ e_s(\alpha) \end{pmatrix}.$$

Hence

$$\{\alpha \in \overline{\mathbb{Q}} : f(\alpha) \in \overline{\mathbb{Q}}\} =$$
$$\{\alpha \in \overline{\mathbb{Q}} : u_0(\alpha) = 0 \text{ and } \exists (\beta, 1, 0, \ldots, 0) \in \overline{\mathbb{Q}}^{s+1} \cap \text{left kernel}\, \mathcal{M}(\alpha)\} \cup \{0\}. \tag{4}$$

• Beukers constructs the matrix $\mathcal{M}(z)$ by desingularization of (3). The order of the poles are not necessarily reduced at each step of his procedure, they can even increase! But they eventually disappear in the final step.

Properties used: 1) the finite non-zero singularities of a non-zero minimal operator that annihilates an E-function are apparent (André 2000).
2) If an E-function F and $\alpha \in \overline{\mathbb{Q}}$ are s.t. $F(\alpha) \in \overline{\mathbb{Q}}$, then $\frac{F(z)-F(\alpha)}{z-\alpha}$ is an E-function (Beukers 2006).

• The set on the RHS of (4) can be explicitly computed. The algorithm stops here.

Example 1

Consider the transcendental E-function

$$f(z) = \sum_{n=0}^{\infty} \frac{n^2\binom{2n}{n}}{(n+1)^2} \frac{(z/2)^{n+1}}{n!}.$$

$$L_{min}: \quad f'''(z) + \frac{1-2z-2z^2}{z(1+z)} f''(z) - \frac{1+4z+z^2}{z^2(1+z)} f'(z) = 0. \qquad (5)$$

L_{inhom} is either (5) or is of order 2.

The differential system

$$Y'(z) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -\frac{1+4z+z^2}{z^2(1+z)} \\ 0 & -1 & \frac{1-2z-2z^2}{z(1+z)} \end{pmatrix} Y(z)$$

has the non-zero solution

$$Y(z) = \left(1, \frac{(1-z)(1-z+2z^2)}{z(1+z)}, \frac{(1-z)^2}{1+z}\right).$$

Hence,

$$L_{inhom}: \quad f(z)+\frac{(1-z)(1-z+2z^2)}{z(1+z)}f'(z)+\frac{(1-z)^2}{1+z}f''(z)=\frac{1}{2}. \quad (6)$$

$u_0(z)=z(z-1)^2$.

Here, it is not necessary to compute Beukers' matrix $\mathcal{M}(z)$. Put $z=1$ in (6): we obtain $f(1)=\frac{1}{2}$.

Conclusion: $f(\alpha)\notin\overline{\mathbb{Q}}$ for any $\alpha\in\overline{\mathbb{Q}}\setminus\{0,1\}$, and $f(1)=\frac{1}{2}$.

$$f(z)=\frac{1}{2}+(z-1)\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{n-1}n!}z^n.$$

Example 2

The roots of u_0 are not always exceptional values for f.

Given two distinct integers $a, b \geq 1$, set $f(z) = z^a e^{az} + z^b e^{bz}$.

$$L_{min}: \quad f''(z) + \frac{1-(a+b)(1+z)^2}{z(1+z)} f'(z) + \frac{ab(1+z)^2}{z^2} f(z) = 0.$$

$L_{inhom} = L_{min}$ and $u_0(z) = z^2(1+z)$.

Hence $f(\alpha) \notin \overline{\mathbb{Q}}$ for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, -1\}$.

$f(-1) = (-1)^a e^{-a} + (-1)^b e^{-b}$by the Lindemann-Weierstrass Theorem.

Hence there is no exceptional $\alpha \neq 0$ for f.

But $f'(-1) = 0$ because $f'(z) = (z+1)(az^{a-1}e^{az} + bz^{b-1}e^{bz})$.