Structure of classical mathematical physics

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Slides of this talk will appear on
http://mech.math.msu.su/~malyshev/doklad_01.pdf
Most results are in the new journal
STRUCTURE of MATHEMATICAL PHYSICS.
New type of the journal - accepts only papers concerning this Project, no anonymous referees.

Mathematics can, possibly even obliged to:

Give qualitative (not quantitative) explanation to many phenomena of physical world, But this explanation should be absolutely rigorous and **structured**

I did not read Hilbert's works but I think that RIGOROUS axiomatics in **global domains** (global means for example that it should include electrodynamics) of mathematical physics never existed. But in many (not all) **small domains** it existed always in the minds of researchers often without exact formulation.

What is Structure -

1. partially ordered set corresponding to one defined in algebra.

2. It is a graph, like a tree but NOT a tree, because vertices can have several incoming edges

3. Each vertex is a container. Highest vertex contains a set of axioms. their number should be **as minimal** as possible. Other vertices contain sets of rigorously proved "theorems", that is **mathematical analogs** of some "physical laws". The more physical "laws" can be derived from axioms - the better. Edges show which higher containers were used for the proof.

4. Looks like pyramids in Egypt. I think now there are thousands of such "Structures-pyramids" in mathematical physics. Moreover, even if the author-mathematician does not say it explicitely, it exists in his mind.

5. Famous example: explosive growth of **Equilibrium** statistical physics - mainly due to only **one axiom - GIBBS field**.

It is now necessary - try to join these small pyramids together, and to analyse "physical pyramids" which do not still exist as CONSISTENT mathematical theories. Main problem - to understand whether Newton motion laws and Maxwell_Loren electrodynamics are consistent or not.

Why only classical physics ?

1. Just to start with something.

2. Only in classical (deterministic) physics there is normal **common sense**, related to our feelings, not just formal algebra.

3. In applications my main interest is in the scales of biophysics, not in super-macro scales of the Universe.

4. Stochasticity will be always present but it should be as minimal as possible.

About quantum and stochastic physics few words later.

AXIOMS

Time R, **space** R^3, **point particles** with **masses** $m > 0$, **charges** q, partially smooth **trajectories** $x(t)$, **momenta** $p = mv = m\frac{dx}{dt}$. Why only point particles - for example, to deduce, as corollaries, the laws of Continuum Mechanics, on any micro, nano and macro scales.

And finally - a **class of FORCES** $F(x, v, t)$ in the Newton law

$$m\frac{d^2x}{dt^2} = F$$

Main question - what kind of forces should be in axioms. Forces can be - **interaction** forces and **external** forces.

About external forces we do not know much. They could be impossible to measure (in particular quite mystical - like soul in human body).

We consider two classes of interaction forces: 1) linear forces, 2) **inverse square** force.

We shall show that even linear forces, though cannot be ultimate axiom, givee sufficiently interesting **pyramid of consequences** for physical world. We will show later that linear forces in some cases give good approximation for theories with Coulomb forces.

1 LINEAR MODELS

Interaction forces Intuitive picture - each particle moves close to the minimum of a small potential well, which also moves somehow. Such well is normally approximated by quadratic function, and nonlinear perturbations will not change qualitative behaviour of the whole system.

We consider general linear system of N_0 point particles jn R^d with $N = dN_0$ coordinates $q_j \in \mathbb{R},\ j = 1, .., N$. Let

$$v_j = \frac{dx_j}{dt}, p_j = m_j v_j,\ j = 1, .., N,$$

$q = (q_1, ..., q_N)^T, p = (p_1, ..., p_N)^T, \psi(t) = (q_1, ..., q_N, p_1, ...$
Potential and kinetic energies are

$$U(\psi(t)) = \frac{1}{2} \sum_{1 \leq j,l \leq N} V_{j,l} q_j q_l = \frac{1}{2}(q, Vq),\ T(\psi(t)) = \sum_{j=1}^{N} \frac{p_j^2}{2}$$

Thus the following system of equations:

$$\ddot{q}_j = -\sum_l V_{j,l} q_l + f(t)\delta_{j,n},\ j = 1, .., N,$$

In the Haniltonian form:

$$\begin{cases} \dot{q}_j = p_j, \\ \dot{p}_j = -\sum_l V_{j,l} q_l + f(t)\delta_{j,n}, \end{cases}$$

or

$$\dot{\psi} = A_0 \psi + f(t) g_n, \qquad (1)$$

where

$$A_0 = \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix} \tag{2}$$

$$g_n = (0, ..., 0, e_n)^T \in \mathbb{R}^{2N},\ e_n = (\delta_{1,n}, .., \delta_{N,n})$$

External Forces We do not know much about them. Normally there are two classes:

1. driving forces. Examples - periodic force like $\sin \omega t$, random stationary process $\xi(t)$;

2. dissipative forces. Example - $\alpha(t)v(t)$ where 1) $\alpha(t) = const < 0$, 2) random point process describing collisions with external particles.

1.1 MAIN RESULTS

NEW approach to famous Boltzman hypothesis We consider the **WORST non-ergodic** (linear system with invariant tori) and show that an external perturbation of ONE only particle provides ergodicity and gives convergence

1) **to Gibbs distribution**

white noise with dissipation ($\alpha > 0$)

$$f = \sigma dw_t - \alpha v(t)$$

Lykov A.A., Malyshev V.A. Convergence to Gibbs equilibrium - unveiling the mystery. Markov Processes and Related Fields, 2013, v.9, № 4.

Lykov A., Malyshev V. A new approach to Boltzmann's ergodicity hypothesis. Doklady RAN. Mathematics), 2015, 92, № 2, p. 624-626.

Convergence to Liouville distribution

For fixed n and random time moments t_k deterministic velocity flips occur

$$v_n(t_k) = -v_n(t_k - 0)$$

Lykov A.A., Malyshev V.A. Harmonic chain with weak dissipation Markov Processes and Related Fields, 2012, v. 8, № 4, p. 721-729.

Lykov A.A., Malyshev V.A. Liouville Ergodicity of Linear Multi-Particle Hamiltonian System with One Marked Particle Velocity Flips. Markov Processes and Related Fields, 2015, v. 21, № 2, pp. 381-412.

Here at random time moments different types of collisions occur.

Lykov A.A., Malyshev V.A. Convergence to equilibrium due to collisions with external particles. Markov Processes and Related Fields, 2018, 24, № 2, 197-227.

Development of chaos Bounded but not sufficiently smooth initial conditions give unbounded chaos as $t \to \infty$.

[2020-02] - Lykov A., Malyshev V. Uniformly Bounded Initial Chaos in Large System Often Intensifies Infinitely. Markov Processes and Related Fields, v 26, № 2, 213-231.

[2020_03] - Lykov A., Malyshev V. How Smooth the System Should be Initially to Escape Unbounded Chaos. Markov Processes and Related Fields, 2020. v 26, № 2, 233-286.

[2020-05] - Lykov A., Malyshev V. When bounded chaos becomes unbounded Proceedings of the XI international conference stochastic and analytic methods in mathematical physics, серия Lectures in pure and applied mathematics (6), место издания Universitätsverlag Potsdam Potsdam, тезисы, 2020, 97-106.

Chaplygin Gas Interesting history. For finite harmonic chain we considered scalings such that particles would not collide. Only this allowed to receive Euler equations Published, and only after this we found that these were Euler equations for Chaplygin gas, which now is very popular in **hundreds of physical papers** and appeared to be related to ball lightning, string theory, dark matter etc.

Lykov A., Malyshev V. From The N-Body Problem to Euler Equations. Russian Journal of Mathematical Physics, 2017, 24, № 1, 79-95.

Resonance phenomena For which ω in external force $\sin \omega t$

$$\max_{0<t<T} |x_k(t)| \to \infty$$

as $T \to \infty$.

Will be published soon.

Local virial theorems Here one can study distribution of kinetic and potential energies in different parts of the big system. For example, one part of the system conserves (potential) energy, another part has big kinetic energy.

2 COULOMB FORCE

Main interaction forces Forces in micro classical physics - only gravitation and electrodynamics.

Gravitation between two point particles is given by **inverse square law**, defined by potential energy

$$U = \frac{C}{|x_1 - x_2|}, C = C_g m_1 m_2$$

There exists great science **Celestial Mechanics** where there is only attraction. But this force is **negligible** between two microparticles and, as external force, is almost constant around human body. And we forget about it.

Electrodynamics consists of **Maxwell equations** (in Gaussian units)

$$\nabla E = 4\pi\rho \tag{3}$$

$$\nabla H = 0 \tag{4}$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t} \tag{5}$$

$$\nabla \times H = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t} \tag{6}$$

and **Lorentz force**

$$F = q(E + \frac{v}{c} \times H) \tag{7}$$

CONSISTENCY problem

Maxwell equations is a classical consistent theory if charges and their velocities are given and we want to find fields E, H. But noone could prove that Maxwell equations and Lorentz force together are consistent. There are many physical papers both pessimistic and optimistic, but all of them are non-rigorous. Main problem - (self-interaction) particle in its own field. Now I will explain similar (but one-dimensional) model, which we proved to be **consistent**.

We consider a system on the real line, consisting of the real scalar field $\phi(x,t), x \in R, t \in R_+$, and point particle with trajectory $y(t) \in R$. The dynamics of this system is defined by two equations: wave equation for the field, «radiated» by the particle,

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} + \beta f(x - y(t)) \quad (8)$$

with the initial conditions

$$\phi(x,0) = 0, \phi'_t(x,0) = 0, \quad (9)$$

and the Newton equation for the particle, driven by its own field,

$$m \frac{d^2 y(t)}{dt^2} = \beta \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \phi(x,t) f(x-y) dx \quad (10)$$

with the initial conditions

$$y(0) = 0, \frac{dy}{dt}(0) = v(0) = v_0 \quad (11)$$

It is well known, see for example [**?**], that for any locally integrable f, and given smooth $y(t)$ the unique solution $\phi(x,t)$ of the linear inhomogeneous equation (8) with initial conditions (9) is also locally integrable and can be written as

$$\phi(x,t) =$$

$$= \frac{\beta}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(x_1 - y(\tau))dx_1 d\tau \quad (12)$$

However the joint system of these equations is nonlinear, and we do not know general rigorous results concerning the structure of its solutions.

Lemma 1 *If the function f is smooth and bounded, then the solution of the system (8)-(11) exists and is unique on all time interval $[0,\infty)$.*

The goal of this paper is to give exact sense and get complete picture of the dynamics for the ultra-local interaction, that is for the case when f is the δ-function.

Theorem 1 *If $f = \delta$ and $|v_0| < c$, then there exists a solution $(\phi(x,t), y(t))$ of the equations (8)-(11) in the domain $x \in R, t \in [0.\infty)$, such that $v(t) = y'(t)$ is a smooth monotone function on $[0, \infty)$. For this solution*

$$\sup_{0 \le t < \infty} |v(t)| < c \tag{13}$$

and $v(t) \to 0$ exponentially fast as $t \to \infty$. Moreover, this solution is unique in the class of smooth solutions, satisfying condition (13).

Energy The equations (8) and (10) can be written in the hamiltonian form

$$\frac{\partial^2\phi(x,t)}{\partial t^2} = -\frac{\delta U}{\delta\phi(x)}, \;\; m\frac{d^2y(t)}{dt^2} = -\frac{\partial U}{\partial y},$$

$$U = U_{ff} + U_{fp} \tag{14}$$

with the formal hamiltonian $H = T_f + T_p + U_{ff} + U_{fp}$, where

$$T_f = \int \frac{1}{2}(\frac{\partial\phi}{\partial t})^2 dx, \;\; T_p = \frac{m}{2}v^2 \tag{15}$$

are the kinetic energies of the field and of the particle, and

$$U_{ff} = \int [\frac{c^2}{2}(\frac{\partial\phi}{\partial x})^2 dx,$$

$$U_{fp} = -\beta \int \phi(x)f(x-y)]dx = -\beta\phi(y)$$

where U_{ff} is the self-interaction energy of the field, U_{fp} is the particle-field interaction.

Theorem 2 *Let $f = \delta$. Then for any fixed t, the supports of the derivatives $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t}$ are bounded in x, and all energy constituents are finite and have the following asymptotics as* $t \to \infty$

$$T_p(t) \to 0, \;\; T_f(t) \sim \frac{\beta^2}{4c}, \;\; U_{ff}(t) \sim \frac{\beta^2}{4c} t,$$

$$U_{fp}(t) = -\frac{\beta^2}{2c} t$$

Interesting conclusion is that there appear fields with energy tending to $+\infty$ and some energy tending to $-\infty$ (possibly like **dark energy**).

COULOMB forces

If in Maxwell equations $\frac{v}{c}$ is very small, we neglect these terms and get only two equations

$$\nabla E = 4\pi\rho, \quad F = qE$$

and, as the corollary, Coulomb (electric) force. Miraculously it is again

inverse square law with $C = C_e q_1 q_2$, but with attraction there is also repulsion. And it seems that dynamics should be much richer than in Celestil mechanics. I was greatly surprised that **Coulomb mechanics** still does not exist (except cases when particles rotate around another, like atom). And first problem is to move this science forward. For example - existence of long molecules and of solid state using **only Coulomb force**.

Why inverse square law is so universal ? There is simple deduction of Coulomb law without Newton axioms, and even **without global space-time**. There are only objects (or particles) with their proper times and interaction between two such proper times (for example, in terms of virtual particles, like in quantum physics).

Malyshev V.A. The Newton and Coulomb Laws as Information Transfer by Virtual Particles. Problems of Information Transmission, 2016, 52, № 3, 308-318.

Малышев В.А. Модели с виртуальными переносчикам взаимодействия в классической физике частиц. Доклады Российской Академии Наук (математическая физика), 2016, том 469, № 3, с. 291-294.

But these are sufficiently difficult non-linear problems, Thus we come back to the Structure where Coulomb law is the highest vertex,

3 COULOMB FORCES in dimension 1

It is important to say that it is NOT the corollary of one-dimensional Laplacian but real inverse square law, assumed in dimension one. We will give simple explanation **why electric current flows.** Up to now existed only Drude model which one can find in any textbook. It is as follows

3.1 Introduction on school level

Consider a circle S of length L. That is the interval $[0, L]$ with identified end points 0 and L. Assume there are N point particles (call them electrons) with mass m and negative charge $q < 0$ at the points

$$0 \leq x_1 < x_2 < \ldots < x_N < L$$

We assume newtonian dynamics with repulsive Coulomb interaction but for simplicity (technical assumption) we assume that only nearest neighbors interact. That is the following equations hold

$$m\frac{d^2x_k}{dt} =$$

$$= \frac{q^2}{(x_k - x_{k-1})^2} - \frac{q^2}{(x_k - x_{k+1})^2} + F(x_k(t), v_k(t), t) \tag{16}$$

Let first $F = 0$. Then, if initially for all k and some fixed v

$$\Delta_k(0) = x_{k+1}(0) - x_k(0) = \frac{L}{N}, \quad v_k(0) = v, \tag{17}$$

the particles will stand still if $v = 0$, that is $x_k(t) = x_k(0)$. If however $v > 0$ then all particles will turn round the circle:

$$x_k(t) = x_k(0) + vt$$

It seems that this could provide eternal energy. But everyone understands that it is a false impression – there is always dissipation of energy. Our devices can take energy and external media can grab the energy. Normally such interaction with external media is modeled on macro scale by the dissipation (friction) force

$$-\alpha v_k(t) \tag{18}$$

with $0 < \alpha$.

On the micro scale this macro force is a consequence of "collisions" with other particles. But in this case, under the same conditions with the dissipation force (18) we will get $v_k(t) \to 0$ for all k. The simplest possibility to avoid this is to add constant force, say $F = qE > 0$. If we define $E = \frac{F}{q}$ it can be imagined as electric field (tension). Or to add potential $U(x) = -\int_0^x E dx$ so that

$$E = -\frac{d}{dx}U(x)$$

Thus, final equations will be

$$m\frac{d^2x_k}{dt} = \frac{q^2}{(x_k - x_{k-1})^2} - \frac{q^2}{(x_k - x_{k+1})^2} + qE - \alpha v_k \tag{19}$$

with the same initial conditions (17). Then, whatever be initial v, for any k as $t \to \infty$

$$v_k(t) \to w = \frac{qE}{\alpha}$$

This is similar to the famous model of electric current proposed by Drude in 1900. This model entered many text books. Note that in Drude's model the particles do not interact. Here they do not interact due to the chosen initial conditions.

Now we can deduce the famous Ohm's law. This is a macroscopic law and we should define the macro variables as the limit of the corresponding discrete quantities as $N \to \infty$. We imagine not discrete point charges but infinitesimally small charges at all points $x \in S$. We refer to the definition of continuum system of point particles in Project 2. So, in each point $x \in S$ there is a "particle" which has trajectory $y(t,x)$ satisfying the equation

$$m\frac{d^2y(t,x)}{dt^2} = qE - \alpha\frac{dy(t,x)}{dt},$$

$$y(0,x) = x, \quad \frac{dy(0,x)}{dt} = 0 \qquad (20)$$

Note that the array (m, q, α) is defined up to a common multiplication factor. It is important that the trajectories of $x_k(t)$ converge to trajectory $y(t, x)$ as $N \to \infty, \frac{k(N)}{N} \to x$.

Anyway the solution is the same as for finite N

$$y(t, x) \to w = \frac{Eq}{\alpha}$$

Introduce the mass and charge densities (scaling correspondingly m and q),

$$\mu = \lim_{N \to \infty} \frac{Nm}{L}, \quad \rho = \lim_{N \to \infty} \frac{Nq}{L}$$

Moreover, α should be scaled correspondingly.

As the force on any particle is $F = qE$ we assume also that the force on any "particle" of the continuum charged medium is $F = \rho E$ and define the potential as follows:

$$U(x) = -\int_0^x E dx = -Ex$$

It is a multivalued (periodic) function on S but, as it is defined only up to additive constant, its derivative (the force) is uniquely defined.

Current $I(x,t)$ at point x at time t is the amount of charge crossing the point x for unit time. It is also a bit puzzling and can be defined only for large N. Namely, in our case for time Δt the amount of charge crossing point x is

$$Q(x, \Delta t) = w \Delta t \rho$$

and we define the current (not depending on x, t due to constant velocity and constant charge density

$$I = \frac{Q(x, \Delta t)}{\Delta t} = w\rho$$

There are two formulations of **Ohm's law**. The first one is the local formulation:

$$I = w\rho = \frac{qE}{\alpha}\rho = \sigma E, \quad \sigma = \frac{\rho q}{\alpha}$$

where σ is called **conductivity**. For the second one define also the potential difference on the interval $[a, b)]$

$$U_{a,b} = U(b) - U(a) = (b - a)E,$$

the **resistivity** $r = \sigma^{-1}$ and **resistance** $R_{a,b} = \int_a^b r dx = (b - a)r$ of the interval $[a, b]$.

Then we have Ohm's law

$$U_{a,b} = (b-a)E = (b-a)I\sigma^{-1} = (b-a)rI = R_{a,b}I$$

What is bad with this model ? Let us see first what famous people say:

"The force pushes the electrons along the wire. But why does this move the galvanometer, which is so far from the force ? Because when the electrons which feel the magnetic force try to move, they push – by electric repulsion – the electrons a little farther down the wire; they, in turn, repel the electrons a little farther on, and so on for a long distance. An amazing thing. It was so amazing to Gauss and Weber – who first built a galvanometer – that they tried to see how far the forces in the wire would go. They strung the wire all the way across the city." This is in Feynman lectures [9], Ch. 16-1.

So, one should consider the case when external force F is not constant but is different from zero only on some interval of the circle with length possibly even much smaller than L (possibly several meters length compared to $L = 100$ kilometers). The simplest set of axioms is the following.

3.2 Axioms

1. We assume 1-dim space and time. Time is R and space is any interval of length L, or the circle of length L, that is the same interval but with identified boundary points.

2. We assume Coulomb interaction with identical particles (equal masses and equal negative charge (like electrons)). One more (technical) axiom – nearest-neighbor interaction. Thus between neighboring particles there is repulsion force. It is likely that this assumption does not influence the qualitative picture we have here.

3. External forces are again of two kinds – dissipative forces $-\alpha v_k$ and deterministic force $F(x), x \in I$, acting on all particles $k = 1, \ldots, N$.

3.3 Phase diagram for static configurations

First of all, the problem of static configuration was considered. Assume that on the interval $[0, L]$ the particles are enumerated as

$$0 \leq x_1 < x_2 < \ldots < x_N \leq L$$

Then if there is no external force, it is clear that in equilibrium the distances between particles will be

$$x_{k+1} - x_k = \frac{L}{N-1} \tag{21}$$

If the external force $F(x)$ is applied, then the most important fact is that the same equality holds but only asymptotically, as $N \to \infty$. Even more important, the new super-micro scale appears as

$$\left| x_{k+1} - x_k - \frac{L}{N-1} \right| = O\left(\frac{1}{N^2}\right) \tag{22}$$

Moreover, a rich phase diagram appears if $F(x)$ depends on N, see [2, 1, 4, 6]. The mentioned results are in some sense for zero temperature.

Quite similar phase diagram was also obtained for Gibbs distribution, see [8], [7].

3.4 Why the Current Flows

Assume now we are on the circle of length L and that initially the particles are at equilibrium, that is the initial velocities are zero and

$$x_{k+1} - x_k = \frac{L}{N}$$

Then it was proved that:

1. almost immediately the particles attain the same constant velocity,
2. they move with this velocity some time at least of the order N
3. during this flow, the density and velocity remain constant, that is Ohm's law holds.

3.5 Further problems

Coulomb networks

1. The circle is the simplest graph with one cycle only. One should prove similar results for more complicated graph. In particular Ohm's law and two Kirchhoff laws.
2. Together with direct current (DC) one could consider micro models of alternative current (AC).
3. Networks with capacities,
4. HodgkinHuxley equations etc.

Moreover, in any biological organism the main force is the Coulomb force. Natural question arises: possibly this force is sufficient to explain many important biological phenomena, including neural networks.

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